HYDRAULIC CHARACTERISTICS OF SAND WAVE IN MOVABLE BED BASED ON STOCHASTIC PROCESSES

SEUNGHWAN SEO
Korea Institute of Civil Engineering and Building Technology, Gyeonggi-Do, Korea, seunghwanseo@kict.re.kr

TADASHI YAMADA
Chuo University, Tokyo, Japan, yamada@civil.chuo-u.ac.jp

ABSTRACT

Based on the stochastic processes, the hydraulic characteristics of sand wave in riverbed were investigated. The irregularity of sediment transportation on the movable bed to have the physical significance have to be dealt with stochastic process. With using the distribution function of the step-length, we derive a new sand wave equation and compare with another equation derived from the deterministic perspective based on principle of hydraulics in movable bed. We can presume higher orders Fokker-Planck equation in stochastic processes and apply sediment transportation. This study aims to analyze the correlation of sand wave equations from different perspectives and to provide mathematical interpretations of them.

Keywords: Stochastic process, Fokker-Planck equation, Step-length, Sand wave

1. INTRODUCTION

The physical quantities we observe on a daily basis are observed as average values of microscopic physical quantities that move randomly on the molecular scale. In other words, the observed physical quantity can be said to be a function having a degree of freedom on the order of the Avogadro's number of a particle, and the particle constantly moves intensely due to thermal motion. In addition, the fluctuation may affect the actual observables. It is indispensable to discuss the concept of stochastic analysis not only on a small molecular scale but also on physical quantities that treated in the fields of hydrology and rivers. To understand natural phenomena by statistical mechanics, when expanding the characteristics at the micro scale to the macro scale, instead of describing all the motions for every single particle, they aggregated into physical quantities by probability distribution. It is common to find the behavior of it. On the other hand, in the moving bed phenomenon, there are various riverbed forms depending on the scale, and the minimum unit that constitutes the form can be classified from the viewpoint of the riverbed wave scale and the viewpoint of the sand particle scale. Furthermore, they are different between determinism and probability theory. Since sediment transport is a phenomenon with discrete and stochastic characteristics, it is necessary to consider the non-equilibrium in the sediment transport process at any scale. Based on these characteristics, modeling is proceeding in consideration of both determinism and probability theory for sediment transport from different scale perspectives, and comparing each role with actual phenomena. However, when viewed on the sand particle scale, the sediment moves randomly, but it does not mention why certain shapes are created in sand wave scale. In other words, when magnifying information from a single particle, the relationship between the two scales is still unclear. Based on these facts, this study aims to connect the sand particle scale with the riverbed wave scale by theoretically describing the relevance of different scale perspectives.

2. DERIVATION OF SAND WAVE EQUATION BASED ON STOCHASTIC PROCESS

The stochastic process of sediment transport is based on the concept of a long rest time "rest period (reciprocal of average pick-up rate)" and instantaneous position change "step-length" proposed by Einstein (1937). Considering the effect of the non-equilibrium of the sediment transport, the continuous equation of the sediment transport rate q and the riverbed height η in consideration of the porosity is expressed by Eq. (1).

$$\frac{\partial \eta}{\partial t} + \frac{\partial q}{\partial x} = 0 \tag{1}$$

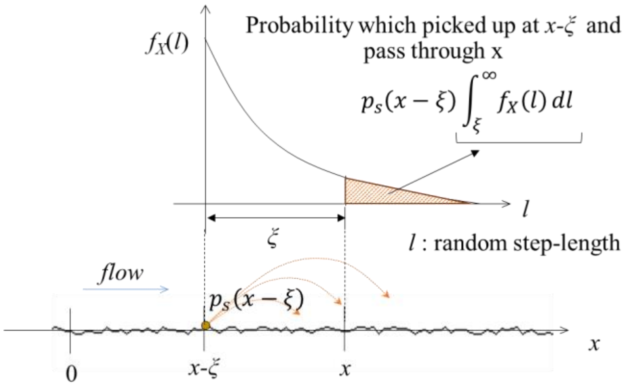

Figure 1. Schematic diagram of sediment movement on the riverbed

Consider the movement of sand grains on the moving bed shown in Figure 1. Considering the number of sand particles picked up by x-ξ and passing through x, the amount of sediment transport can be expressed as Eq. (2) using the pick-up rate p_s.

$$q(x) = \int_0^x \frac{1}{A_2 d^2} p_s(x-\xi) A_3 d^3 \int_{\xi}^{\infty} f_X(l) dl d\xi \tag{2}$$

Here, $f_X(l)$ is the probability density function of the step-length, and follows the exponential distribution as shown in Figure 1. A_1, A_2, A_3 are the shape coefficients of each dimension, $c_0 = A_1 d$. Eq. (3) can be obtained by changing Eq. (2) using the Leibniz integral rule.

$$\frac{\partial \eta}{\partial t} = c_0 \left\{ p_d(x) - p_s(x) \right\} \tag{3}$$

On the right side of Eq. (3), p_d is a deposit rate that indicates the amount that is picked up from the point of x-ξ, moves about ξ, and stops at x. Regarding the pick-up rate p_s, the relationship of Eq. (4) is established by the experimental results of Nakagawa and Tsujimoto (1979). From this relationship, it can be seen that the pick-up rate and the bottom shear stress are proportional.

$$p_s \simeq 0.03\tau_* \sqrt{d / \{(\sigma/\rho - 1) g\}} \qquad p_s \simeq e\tau_* \tag{4}$$

The shear stress in bottom surface is expressed as Eq. (5) proposed by Hayashi (1970).

$$\tau_* = \beta_1 \left(1 + \alpha \frac{\partial \eta}{\partial x}\right) u_0^2 \qquad u_0 = a_1 + b_1 \eta \tag{5}$$

Hayashi (1970) showed that it is important to reflect the effect of the local gradient of the riverbed on the phase shift that occurs between the shear stress and the riverbed shape. The parameter representing the effect is α. The local velocity u_0 uses the perturbed velocity component, and the perturbation component is proportional to the first order of the riverbed height. Eq (6) and (7) expresses the pick-up rate.

$$p_s(x) = a + b\eta + a\alpha \frac{\partial \eta}{\partial x} \tag{6}$$

$$p_s(x-\xi) = a + b\eta(x-\xi) + a\alpha \frac{\partial \eta(x-\xi)}{\partial x} \tag{7}$$

$$a = e\beta_1 a_1^2 \quad b = 2ea_1 b_1 \beta_1 \tag{8}$$

When the Eq. (6) and (7) are Taylor-expanded and substituted into the Eq. (3). Using the following Eq. (9), Eq. (10) is obtained.

$$I_n = \int_0^{\infty} \xi^n f_X(\xi) d\xi \tag{9}$$

$$\frac{\partial \eta}{\partial t} + c_0 b I_1 \frac{\partial \eta}{\partial x} + c_0 \left(a\alpha I_1 - \frac{bI_2}{2}\right)\frac{\partial^2 \eta}{\partial x^2} + c_0 \left(\frac{bI_3}{6} - \frac{a\alpha I_2}{2}\right)\frac{\partial^3 \eta}{\partial x^3} + c_0 \left(\frac{a\alpha I_3}{24} - \frac{bI_4}{6}\right)\frac{\partial^4 \eta}{\partial x^4} = 0 \tag{10}$$

Here, in means the nth moment of the probability density function $f_X(\xi)$ of the step-length, and the finally obtained Eq. (10) is a small-scale riverbed wave equation using the probability density function of the step-length. The terms on the left side of this equation represent the advection, diffusion, dispersion and dissipation effects, which depend on the moment of the step-length probability density function. The moment of each order represents the mean, variance, skewness, and kurtosis. The first term on the left is an unsteady term, and the second is an advection term. This term is determined by the average value of step-length. The third term is the diffusion term, and the diffusion coefficient appears as the difference between the variance and the expected value. In addition, a negative diffusion coefficient means instability of the system, which is a condition for the generation of riverbed waves. Therefore, the larger the riverbed gradient effect α and the flow velocity, the higher the deposit rate, and the larger the sand waves grow. Become. The fourth term is the variance term, and the variance effect expressed by the difference between skewness and variance. The fifth term is the dissipation term, which expressed by the difference between skewness and kurtosis. The advection of sand waves is deterministic and the effects of diffusion, dispersion and dissipation are determined by deterministic physical parameters and moment functions. That is, the irregularity of sediment transport represented by the appearance of moments up to the fourth order, and the sediment transport information on the upstream side is transmitted to the downstream side with a certain probability according to the exponential distribution of the step-length. If the probability density function of the step-length is not a distribution function but a delta function, the step-length is deterministic. From such a deterministic point of view, Yamada and Ikeuchi (1987) proposed a small-scale riverbed wave equation based on hydraulic principles.

$$\frac{\partial \eta}{\partial t} + \{m + m(m-1)\eta\}\frac{\partial \eta}{\partial x} + \left[\alpha - \delta\frac{m}{h}\{1+(m-1)\eta\}\right]\frac{\partial^2\eta}{\partial x^2} + \left[\frac{1}{2}\delta^2\frac{m}{h^2}\{1+(m-1)\eta\} - \alpha\frac{\delta}{h}\right]\frac{\partial^3\eta}{\partial x^3} + \frac{1}{2}\alpha\frac{\delta^2}{h^2}\frac{\partial^4\eta}{\partial x^4} = G(x) \qquad (11)$$

where, α: riverbed gradient effect, δ: average step-length, m: proportional coefficient of sediment transport equation

Eq. (11) is the governing equation for riverbed waves derived from a deterministic point of view, and is in the form of a nonlinear partial differential equation. This equation consists of deterministic parameters for advection, diffusion, dispersion and dissipation effects. In this equation, the first term on the left side is an unsteady term related to riverbed height, and the second term is an advection term. This term causes the non-linearity of the riverbed wave, which results in a sloping riverbed form. The third term is the diffusion term. If the diffusion coefficient takes a negative value, the riverbed wave becomes unstable, meaning the generation of the riverbed wave. The diffusion coefficient is expressed by the difference between the two parts, and the effect of shifting the distribution of bottom shear stress on the riverbed shape, or the effect of α on the effect of riverbed gradient and the effect of sediment transport lagging upstream. The δ represents the magnitude of the two effects determines the generation of riverbed waves. In other words, the diffusion coefficient is a condition for generating riverbed waves. The fourth term is the variance term, which indicates that the forward tilt of the sand wave is dispersed by the occurrence of short wavelengths. The fifth term is the dissipative term, which acts to suppress the growth of the riverbed wave itself, meaning the effect of restoring the instability of the system due to negative diffusion.

3. DISCUSSIONS

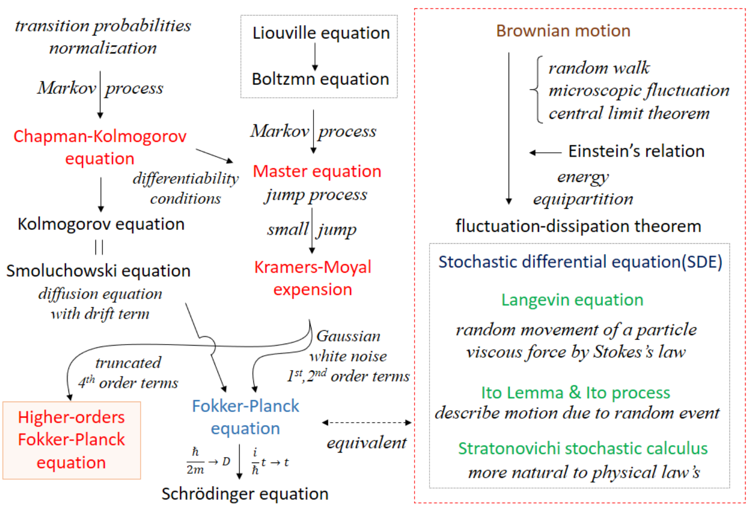

Figure 2. Structure of basic formula in stochastic process theory (Van kampen, 2007)

Figure 2 shows the structure of various basic equations used when dealing with physical quantities in consideration of stochastic theory. There is a close relationship among the equations in a variety of approaches. Here, instead of understanding the stochastic process of one random variable, the Chapman-Kolmogorov equation using the Markov process and the transition probability transformed in differential form to observe the change in the density distribution of the random variable, and the master equation is obtained. The formula represents a discrete stochastic phenomenon that follows a Poisson process called a jump process, and means jump of a random variable. If jump is small, the Kramers-Moyal equation is obtained. The characteristics of random variables in such a stochastic process are considered in the case of sediment transport. Einstein (1950) proposed a zig-zag model based on the Poisson process for bed load motion in the moving bed phenomenon, and found that the step-length of sand particles caused by flowing water behaves exactly like jump in the master equation. Therefore, if the pick-up rate is completely white noise-like, the diffusion type Fokker-Planck equation will be obtained. , Which corresponds to the higher-order Fokker-Planck equation. As shown in Figure 2, if the Kramers-Moyal expansion expanded to the fourth order without using white noise, a higher-order Fokker-Planck equation such as Eq. (12) is obtained.

$$\frac{\partial P(x,t)}{\partial t} = -a_1 \frac{\partial P}{\partial x} + \frac{1}{2} a_2 \frac{\partial^2 P}{\partial x^2} - \frac{1}{6} a_3 \frac{\partial^3 P}{\partial x^3} + \frac{1}{24} a_4 \frac{\partial^4 P}{\partial x^4}$$
$$a_n = \int l^n f(l)\, dl \tag{12}$$

Here, a_n is determined by giving a distribution function. In the case of sediment transport, a_n corresponds to a step-length distribution. The Eq. (10) is derived from the sand particle scale and Eq. (12) is derived from the micro scale, which is a random variable, and the jump phenomenon of the random variable is regarded as the saltation phenomenon of sand particles. The Fokker-Planck equation is a type of coarse-graining that describes the motion of many particles from the Langevin equation that describes the motion of a single Brownian particle. If the time scale is coarse-grained with such coarse-graining, it becomes a Markov process. Therefore, the small-scale riverbed wave equation using the distribution function in Section 2 is an equation having characteristics of Markov process like the motion of Brownian particles.

4. CONCLUSIONS

In this study, a small-scale riverbed wave equation using a step-length distribution function was derived. When the random characteristics of the sediment transport expressed as moments and expressed again as expected values in terms of quantities determined deterministically, the generation of riverbed waves by a mechanism similar to the deterministic equation based on hydraulic principles. In addition, from the viewpoint of stochastic process theory in statistical mechanics, the jump process of random variables can be regarded as saltation motion of sand particles, and corresponds to the higher order Fokker-Planck equation. This study provides a mathematical basis for explaining the relationship between the different scales and coarse-graining when explaining the sand wave scale from the sand particle scale in sediment transport. This can be considered as a general theory of pattern formation in sand wave.

REFERENCES

Einstein, H. A. (1937). Der Geschiebetrieb als Wahrscheinlickeits-Problem, Mitteilung der Versuchsanstalt fur Wasserbau und der Eidgenössische Technishe Hochschule in Zurich, *Verla Rascher*, pp.110

Einstein. H. A. (1950). The Bed-Load function for Sediment Transportation in Open Channel Flow, *USDA, Soil Conservation Service, Technical Bulletin*, No.1026, pp.1-71.

Hayashi, T. (1970). Formation of Dunes and Antidunes in Open channels, *Proc. ASCE.*, Vol. 96, No. HY2, pp.431-480.

Nakagawa, H., and Tusjimoto, T. (1979). Disturbance process of sand surface in movable bed due to movement of gravel (in Japanese), *Journal of JSCE*. Vol. 291, pp.73-83

Van kampen, N. G. (2007). *Stochastic processes in physics and chemistry*, NORTH-HOLLAND.

Yamada, T. and Ikeuchi, M. (1987). Study on the mechanism of ripple in riverbed (in Japanese), *Proc. Hydraulic Lecture papers*