RAINFALL-RUNOFF ANALYSIS CONSIDERING THE UNCERTAINTIES OF RAINFALL INTENSITY AND MODEL PARAMETERS BASED ON THE THEORY OF STOCHASTIC DIFFERENTIAL EQUATION

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ABSTRACT

Serious disasters with enormous damage can change the common recognition of the standard of disaster prevention plans. For example, The Kanto-Tohoku heavy rainfall in September 2015 caused a severe flood disaster in the Kinugawa river basin. Before these severe disasters that occurred in recent years. The Basic standard of disaster prevention plans is to see if a plan can respond to a designed force of certain return period. This is usually called structural countermeasures. However, after these disasters, society now recognized that both structural countermeasures and non-structural countermeasures are necessary. On the other hand, unlike earthquakes and tsunami, there is usually enough time for residents to evacuate in flood disasters if they are appropriately informed. Thus, the prediction of runoff is a critical index for evacuation. To make the prediction, it needs to consider the uncertainty of rainfall intensity and model parameters in the rainfall-runoff analysis. Besides, how to consider the basic nature of uncertainty in the rainfall-runoff system had always been an important topic in hydrology. M.Hino(1974) had first introduced the Kalman filter in forecasting the rainfall-runoff process which considered the uncertainty of the process, since then methods such as Kalman filter, ensemble Kalman filter, particle filter, data assimilation, had been used to consider the uncertainty effects in the rainfall-runoff process. However, these methods are based on filtering theory and statistical methods, which cannot recognize the physical meaning of the uncertainty. The present study is based on the theory of stochastic differential equation, aimed at suggesting a new way of rainfall-runoff analysis which can not only consider the uncertainty in the system but also identify the physical meaning of these uncertainties

Keywords: *Rainfall-runoff analysis , Stochastic differential equation , Uncertainty of model parameters*

1. INTRODUCTION

The present study aimed at suggesting a rainfall-runoff analysis that can consider the uncertainties of rainfall and model parameters based on the theory of stochastic differential equation. In the second section, we will introduce a physical-based, deterministic rainfall-runoff model. The parameters of this model have clear physical meanings so that the uncertainties of the characteristic of the river basin can be directly concerned. In the third section, we will introduce how to consider the uncertainties of rainfall density and parameters using the theory of stochastic differential equation.

2. DETERMINISTIC RAINFALL-RUNOFF ANALYSIS

A physical-based, deterministic rainfall-runoff model was proposed by S.Kure and T.Yamada(2009) . This model is a simplification of the continuity equation and the momentum equation of the rainfall-runoff process on a mountain slope.

$$\frac{dq}{dt} = a_0 q^{\beta}(r(t) - q) \tag{1}$$

$$\alpha = \frac{k_s i}{D^{\gamma-1} w^{\gamma}} \quad \beta = \frac{m}{m+1} \quad a_0 = \frac{\beta}{1-\beta}\left(\frac{\alpha}{L}\right)^{1-\beta} \tag{2}$$

Eq. (1) is the basic equation. a_0 and β are parameters directly determined by the physical properties (soil quality, slope gradient, etc.) of the basin, which shown in Eq.(2). i is the slope gradient, k_s is the coefficient of permeability of saturated soil[cm/sec], D is the thickness of Permeable layer[cm], γ is a non-dimensional parameter which represents the permeability of the soil, w is the effective porosity, and L is the length of the slope[m]. q (t) is the runoff rate [mm / h], which is the flow rate at the dominant section of the basin divided by the basin area. r (t) is the basin average effective rainfall intensity [mm / h]. The concept of the basic equation is shown in Figure 1. When the effective rain intensity r (t) is substituted into the equation, it can be solved, and the time series of runoff rate, discharge, or water level at the dominant section of the basin can be obtained. However, since it is difficult to predict rainfall intensity every moment, and since the physical properties of the basin may vary over time, it is necessary to divide the rainfall intensity and parameters into deterministic (currently predictable) part and uncertain (currently difficult to predict) part.

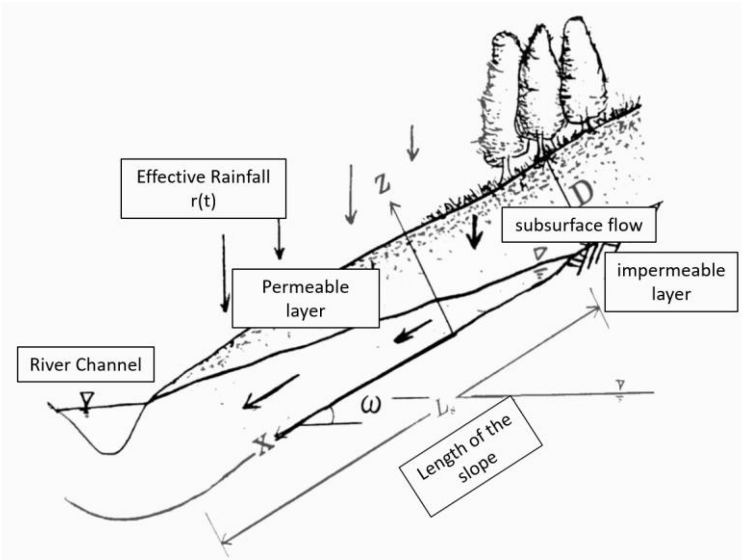


Figure 2. The concept of the rainfall-runoff process on a mountain slope

3. RAINFALL-RUNOFF ANALYSIS BASED ON THE THEORY OF STOCHASTIC DIFFERENTIAL EQUATION

3.1 How to consider the uncertainty of rainfall intensity

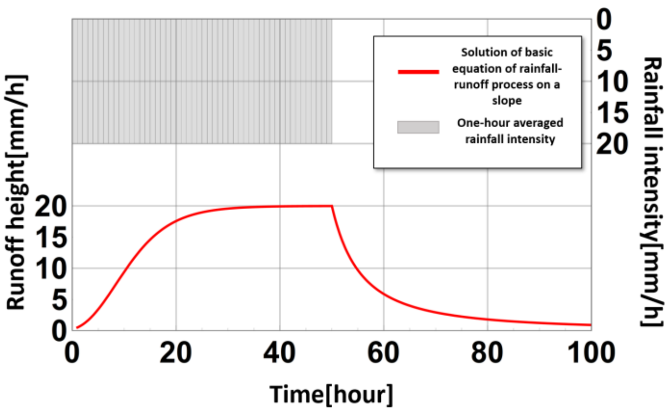
To consider the uncertainty of rainfall intensity, we can first separate the rainfall into $\bar{r}(t)$ and $r'(t)$ two parts,

$$\frac{dq}{dt} = a_0 q^{\beta}(\bar{r}(t) + r'(t) - q) \tag{3}$$

As $r'(t)$ is the uncertainty part of effective rainfall, which is a random variable, Eq.(3) became a differential equation that cannot be solved directly. Acrodding to K.Yoshimi, Y.Yamada's(2016) work, by applying the theory of stochastic differential equation, which suggested by K.Ito(1946), we can prove that the probability density function of q follows the following Fokker-Planck type equation.

$$\frac{\partial P(q,t)}{\partial t} + \frac{\partial a_0 q^{\beta}(\bar{r}(t) - q)P(q,t)}{\partial q} = \frac{1}{2}\frac{\partial^2 \left(a_0 q^{\beta}\sigma\sqrt{T_L}\right)^2 P(q,t)}{\partial q^2} \tag{4}$$

a_0 and β are the same parameters as in Eq.(1), $P(q,t)$ is the probability density function of q over time, σ is the standard division of $r'(t)$, and T_L is the interval which self-correlation function of $r'(t)$ goes to zero. By solving Eq.(4), we can get the time



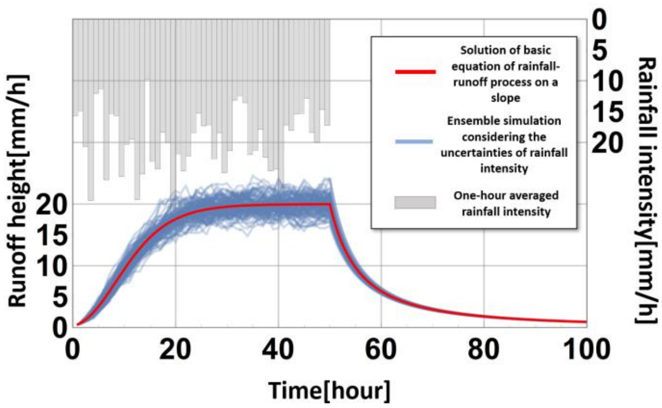


Figure 2. Solution of the basic equation of rainfall-runoff system in slope（left：Not considering rainfall uncertainty, right：Considering rainfall uncertainty）

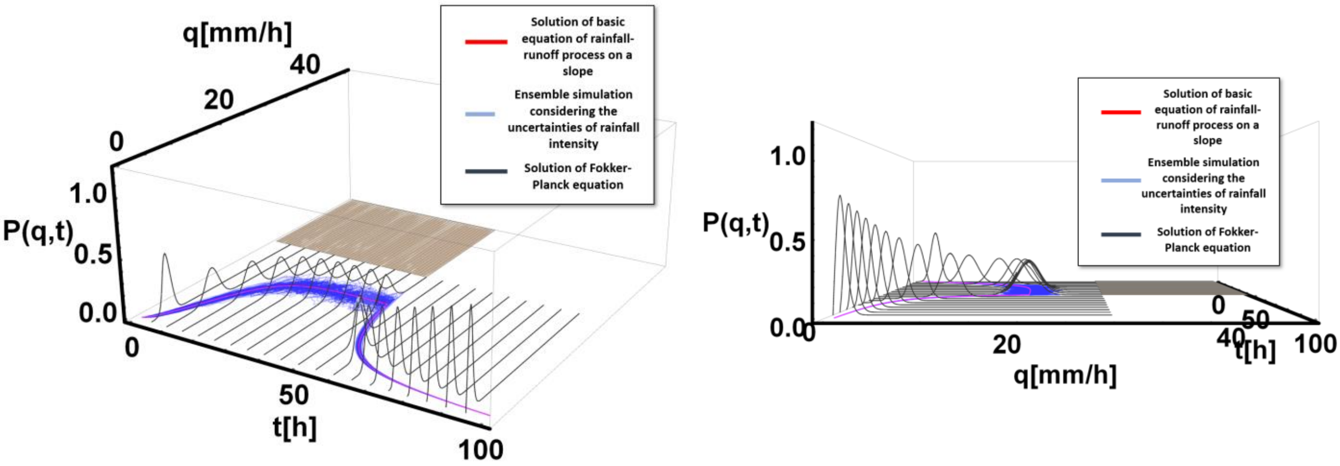


Figure 3. The time evolution of the probability density function of runoff rate (Fokker-Planck equation's solution)

3.2 How to consider the uncertainty of Parameters

If we directly applied the method mentioned in the previous section, the parameters will also need to be divided into two parts, like rainfall intensity. However, the properties of the parameters' uncertainty and the rainfall's uncertainty are quite different. The deterministic component and the uncertain component correspond to the fluctuation on the long-time scale and the fluctuation on the short-time scale, respectively. In the case of rainfall, the fluctuation component on the long-term scale becomes the time average of the observed values, and the uncertainties of the observed values at each time are canceled out by the effect of the average. However, the fluctuation component on the short-time scale has greater uncertainty. On the other hand, since the parameters represent the physical properties of the basin, they are relatively stable, so that the fluctuation on the short-time scale is small. However, the fluctuation components on the long-time scale cannot be directly observed. Therefore, when considering the uncertainty of the parameters, it is necessary to include the uncertainty of the fluctuation component on the long-term scale in the Fokker-Planck equation.

Since the fluctuation component of the short-time scale is treated as a random external force of the system, it is not included in the independent variable of the probability density function of the system variable. In Eq.(4), the independent variables for P are q and t. However, when considering the uncertainty of the fluctuation component on a long time scale, the variable cannot be treated as an external force and becomes an internal variable of the system. Therefore, when considering the uncertainties of the parameters a_0 and β in Eq.(1), the independent variables of the probability density function are, q, α, β, and t. When describing a physical system with simultaneous ordinary differential equations, if the number of equations does not match the number of internal variables, the system will not be closed, so equations that describe the time evolution of a_0 and β will also be required. Therefore, Eq.(1) becomes the following Eq.(5):

$$\begin{cases} \frac{dq}{dt} = a_0 q^{\beta}(r(t) - q) \\ \frac{da_0}{dt} = 0 \\ \frac{d\beta}{dt} = 0 \end{cases} \tag{5}$$

For simplicity, considering only the uncertainties of the fluctuation components on the long time scale of the parameters a_0 and β, the Fokker-Planck equation corresponding to Eq.(5) is as follows.

$$\frac{\partial P(q, a_0, \beta, t)}{\partial t} + \frac{\partial a_0 q^{\beta}(\bar{r}(t) - q) P(q, a_0, \beta, t)}{\partial q} = 0 \tag{6}$$

Comparing Eq. (4) and Eq. (6), the right side of Eq. (6) is 0 because short time scale fluctuation components of a_0 and β are not considered. Although the left side of both equations is similar in form, it is not the same equation because the independent variables of P are different. Eq. (4) cannot consider the uncertainties of a_0 and β, while Eq. (6) can treat the probability distribution of a_0 and β as the initial condition. Finally, by integrating the solution P (q, a_0, β, t) of Eq. (6) with respect to a_0 and β, the probability density function P (q, t) of the runoff hight q is obtained.

In order to discuss the properties of Eq. (6), we used the following rainfall event as an example. The target basin is the Kusaki Dam basin in Japan. Kusaki Dam is a dam constructed in the upper stream of the Watarase River in the Tone River system. The basin area controlled by the dam is about 254 km2, and the outline of the basin is shown in Figure 4.

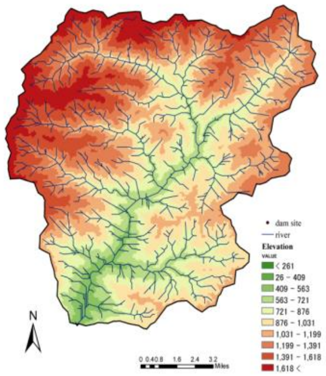


Figure 4. The basic information about Kusaki dam river basin

The flood event selected was from August 8, 2003 to August 12, 2003. The total rainfall is about 250mm. The calculation conditions are listed as follows: Not considering the uncertainties of rainfall intensity. Assuming that the uncertainties of the model parameters a_0 and β follow a normal distribution, set a_0 , β to $N(0.045, (k * 0.045)^2), N(0.41, (k * 0.41)^2)$, where N is the normal distribution, and k is a coefficient that represents the standard deviation as a percentage to the average value. In the present study, k = 10%, 20%, and 30%,3 cases were calculated.

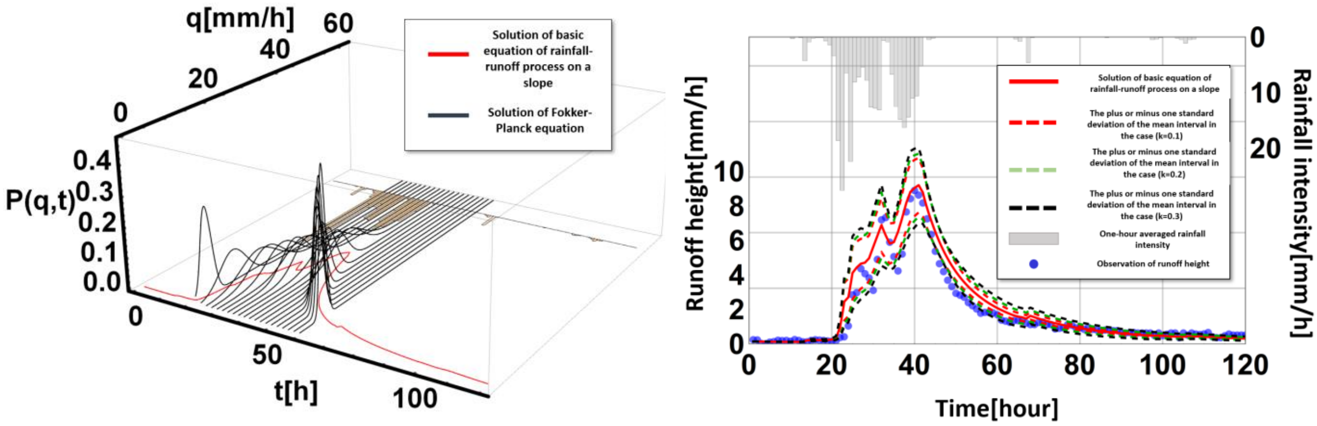


Figure 5. Rainfall-runoff analysis of 2003-08-08 rainfall event in Kusaki dam river basin, considering the uncertainty of model parameters

Figure 5 shows the analysis results. The left side figure shows the solution of the Fokker-Planck equation directly, and right side figure shows the range of one standard deviation above and below the mean value and the observation of runoff. If the distribution of the runoff height q can be approximated by the normal distribution, the probability that the observed value falls into the range shown in the figure will be about 68%, so this range can also be considered as the prediction interval with 68% reliability. As shown in Fig. 5, the prediction range near the peak was the widest, and its width was about 25% above and below the average. On the other hand, comparing the 3 cases(k = 10%, 20%, and 30%), it was found that even if the standard deviation of the parameter increased, the prediction interval hardly changed.

4. CONCLUSIONS

The present suggested a rainfall-runoff analysis method based on the theory of stochastic differential equation. Uncertainty of rainfall and model parameters had been considered, and the control equation(Fokker-Planck type equation) of the probability density function of runoff height had been derived. This equation can be used to discuss how the uncertainties in the rainfall-runoff process affect the final result and to identify the over flow risk in flood events.

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